

# Constructing large tables of numbers of maps by orientable genus

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## Abstract

The Carrell-Chapuy recurrence formulas dramatically improve the efficiency of counting orientable rooted maps by genus, either by number of edges alone or by number of edges and vertices. This paper presents an implementation of these formulas with three applications: the computation of an explicit rational expression for the ordinary generating functions of rooted map numbers with a given positive genus, the construction of large tables of rooted map numbers, and the use of these tables, together with the method of A. Mednykh and R. Nedela, to count unrooted maps by genus and number of edges and vertices.

## 1 Introduction

A *map* is a 2-cell imbedding of a connected graph, loops and multiple edges allowed, on a compact surface, which in this article will be taken to be orientable and without boundaries, and is thus characterized by a single non-negative integer, its *genus*. A map is *rooted* if a *dart* – an edge-vertex incidence pair – is distinguished as the root. By *counting* maps we mean counting equivalence classes of maps under orientation-preserving homeomorphism; in the case of rooted maps, the homeomorphism must preserve the distinguished oriented edge. In this case the homeomorphism preserves all the darts [15], so that rooted maps can be counted without considering the symmetries of the maps, which is why rooted maps were counted before unrooted maps.

Let  $m_g(n)$  be the number of rooted maps with  $n$  edges on the orientable surface of genus  $g$ . Let  $M_g(z) = \sum_{n \geq 0} m_g(n)z^n$  be the ordinary generating function counting genus- $g$  rooted maps by number of edges (the exponent of  $z$ ). Let  $m_g(v, f)$  be the number of rooted genus- $g$  maps with  $v$  vertices and  $f$  faces. By face-vertex duality, this number is equal to the number  $m_g(f, v)$  of rooted genus- $g$  maps with  $f$  vertices and  $v$  faces. The ordinary generating function that counts rooted genus- $g$  maps is the following formal power series in two variables  $u$  and  $w$ :

$$M_g(w, u) = \sum_{v, f \geq 1} m_g(v, f)w^v u^f. \quad (1)$$

The Carrell-Chapuy recurrence formulas [6] dramatically improve the efficiency of counting rooted maps by genus. We show how to use them to determine explicit rational expressions for the generating functions  $M_g(z)$  and closed-form formulas for the numbers  $m_g(n)$ . We also have used Carrell-Chapuy recurrence formulas to construct large tables of numbers of rooted and unrooted maps of genus up to 50 with up to 100 edges. Our goal is to provide these numbers to researchers for further studies of their properties.

The paper is organized as follows. Section 2 summarizes the history of two closely related problems, namely computing numbers of rooted maps by genus and finding a closed form for their generating functions. Section 3 presents formulas for numbers of rooted maps with a fixed genus. In Section 4 we discuss counting unrooted maps and in Section 5 we give a complexity analysis and the results of time trials. In the appendix we include a table of numbers of unrooted maps counted by genus, number of edges and number of vertices. We do not include a table of numbers of rooted maps because the reader can easily construct such a table from the recurrence in [6, Corollary 3] or the optimized version of it that we present as formula (4) here. A larger table, a table of numbers of rooted maps and a text file of the source code are available from the second author on request and can be found in release 0.4.0 of the MAP project [9]. The source code can also be found in [20].

## 2 Historical notes

For counting by number of edges alone, W. T. Tutte [15] first showed that the generating function  $M_0(z)$  for rooted planar maps can be parametrically defined by  $M_0(z) = (3 - \xi)(\xi - 1)/3$ , where the parameter  $\xi$  is the series in  $z$  satisfying  $\xi = 1 + 3z\xi^2$ . Tutte also found a closed-form formula for  $m_0(n)$ . For counting with two parameters (i.e. by number of edges and vertices, edges and faces, or vertices and faces), W. T. Tutte [16] and D. Arquès [1, Theorem 4] respectively found a parametric polynomial definition of  $M_0(w, u)$  and a parametric rational definition of  $M_1(w, u)$ . Arquès also obtained a closed-form formula for the number of rooted toroidal maps with  $n$  edges and another one for the number of rooted toroidal maps with  $v$  vertices and  $f$  faces.

In [16], a recursive formula was found for the number of rooted planar maps given the number of vertices, the number of edges, and the degree of the face containing the root; these numbers of maps were then added over all possible degrees of this face and the result expressed in terms of generating functions. In [17], this method was generalized to obtain a recursive formula for the number of maps of genus  $g$  with a distinguished dart in each vertex given the number of vertices and the degree of each one; these numbers were then multiplied by the appropriate factor and added over all possible non-increasing sequences of vertex-degrees summing to  $2n$  to obtain the number of rooted maps of genus  $g$  with  $n$  edges and  $v$  vertices. A table of these numbers of maps with up to 14 edges appears in [17] (see [23] for a published account of this work and a table of maps with up to 11 edges) but no attempt was made there to express this result in terms of generating functions. In [3] an improvement on the method of [17] was introduced: to count rooted genus- $g$  maps it is sufficient to know the degree of the first  $g + 1$  vertices and to distinguish a dart of only the first vertex as the root, thus reducing the number of maps that have to be considered.

For any genus  $g$  the existence of a parametric rational expression for the generating functions  $M_g(z)$  and  $M_g(w, u)$  is stated by E. Bender and E. Canfield, in [4] for  $M_g(z)$  and in [5] for  $M_g(w, u)$ . The first of these two papers [4] also presents explicit rational functions for  $M_2(z)$  and  $M_3(z)$ . A common pattern for all these rational functions is proposed, and an upper bound for the degree of their numerator is conjectured. For the univariate function  $M_g(z)$  a bound was found in [7] and refined in [8]. For the bivariate function  $M_g(w, u)$  a bound was proved in [2]. These results are summarized in the following two theorems.

**Theorem 1** ([8]). *For any positive integer  $g$ , the ordinary generating function  $M_g(z)$  counting rooted maps on a closed orientable surface of genus  $g$  by number of edges (exponent of  $z$ ) can be written as*

$$M_g(z) = z^{2g}(1 - 2m)^{2-3g}(1 - 3m)^{-2}(1 - 6m)^{3-5g}P_g(m),$$

where  $m = \frac{1 - \sqrt{1 - 12z}}{6}$  and  $P_g(m)$  is a polynomial of  $m$  of degree at most  $4g - 4$ .

**Theorem 2** ([2, Theorem 1]). *For any positive integer  $g$ , the ordinary generating function  $M_g(w, u)$  counting rooted maps on a closed orientable surface of genus  $g$  by number of vertices*

(exponent of  $w$ ) and faces (exponent of  $u$ ) can be written as

$$M_g(w, u) = \frac{pq(1-p-q)P_g(p, q)}{\left[(1-2p-2q)^2 - 4pq\right]^{5g-3}}, \quad (2)$$

where  $P_g(p, q)$  is a symmetric polynomial in  $p$  and  $q$  of total degree at most  $6g - 6$  with integral coefficients.

In [21] the first and second authors calculated the polynomial  $P_g(p, q)$  for  $g$  up to 6 and thus counted rooted maps of genus up to 6 by number of vertices and faces as well as by number of edges (using Theorem 1). In [22] the first author of that paper, using a more powerful computer, extended these calculations up to genus 10 and also counted unrooted maps of genus up to 10 by number of vertices and faces, the second author counted rooted maps of genus up to 11 by number of edges and the third author, A. Mednykh, counted unrooted maps of genus 11 by number of edges. The cost of counting unrooted maps, once a table of numbers of rooted maps has been constructed, was greatly dominated by the cost of counting rooted maps.

More recently, a far more efficient method for counting rooted maps with a fixed genus was discovered by S. R. Carrell and G. Chapuy [6]. They showed [6, Theorem 1] that the number  $m_g(n)$  of rooted maps of genus  $g$  with  $n$  edges satisfies the following recurrence relation (we have modified the formulas for the sake of computational efficiency):

$$\begin{aligned} (n+1)m_g(n) &= (8n-4)m_g(n-1) \\ &\quad + (2n-3)(n-1)(2n-1)m_{g-1}(n-2) \\ &\quad + 3 \sum_{\substack{i+j=g \\ i,j \geq 0}} \sum_{\substack{k+l=n-2 \\ k \geq 2i, l \geq 2j}} (2k+1)(2l+1)m_i(k)m_j(l) \end{aligned} \quad (3)$$

for  $n \geq 1$ , with the initial conditions  $m_0(0) = 1$  and  $m_g(n) = 0$  if  $g < 0$  or  $n < 2g$ . For counting with two parameters, Carrell and Chapuy showed [6, Corollary 3] that the number  $m_g(n, f)$  of rooted maps of genus  $g$  with  $n$  edges and  $f$  faces satisfies the following recurrence relation:

$$\begin{aligned} (n+1)m_g(n, f) &= (4n-2)(m_g(n-1, f) + m_g(n-1, f-1)) \\ &\quad + (2n-3)(n-1)(2n-1)m_{g-1}(n-2, f) \\ &\quad + 3 \sum_{\substack{i+j=g \\ i,j \geq 0}} \sum_{\substack{k+l=n-2 \\ k \geq 2i, l \geq 2j}} \sum_{\substack{u+v=f \\ u,v \geq 1}} (2k+1)(2l+1)m_i(k, u)m_j(l, v) \end{aligned} \quad (4)$$

for  $n, f \geq 1$ , with the initial conditions  $m_0(0, 1) = 1$  and  $m_g(n, f) = 0$  if  $g < 0$  or  $n < 2g$  or  $f < 1$  or  $n - f + 2(1 - g) < 1$ .

### 3 Fixed genus formulas

This section shows simple – but as far as we know not yet published – consequences of Theorem 1 (about the rationality of the generating series  $M_g(z)$ ) for the computation of rooted map numbers. The first consequence, given in Theorem 4, is a recurrence formula between numbers of rooted maps with the same positive genus  $g$ . The second consequence, given in Theorem 5, is a closed formula for the number  $m_g(n)$ , for any positive genus  $g$  and any number of edges  $n$ . This formula depends on integers which are the coefficients of the polynomial  $P_g(m)$  from Theorem 1. We start this section with a theorem completing Theorem 1 with an explicit relation between this polynomial and the numbers of rooted maps with the same genus and up to  $6g - 4$  edges. In this section  $[x^n]S(x)$  denotes the coefficient of  $x^n$  in the formal power series  $S(x)$ . By convention, a sum over an empty domain is equal to zero.

**Theorem 3.** For any positive integer  $g$ , the ordinary generating function  $M_g(z)$  counting rooted maps on a closed orientable surface of genus  $g$  by number of edges (exponent of  $z$ ) is

$$M_g(z) = z^{2g} P_g(m) / F_g(m), \quad (5)$$

where  $m = \frac{1 - \sqrt{1 - 12z}}{6}$  and  $F_g(m)$  and  $P_g(m)$  are the polynomials defined by

$$F_g(m) = (1 - 2m)^{3g-2} (1 - 3m)^2 (1 - 6m)^{5g-3} \quad (6)$$

and  $P_g(m) = \sum_{l \leq 4g-4} p_{g,l} m^l$  with

$$p_{g,l} = \sum_{n=2g}^{6g-4} (-1)^{l-n} m_g(n) \sum_{i+j+k=l-n+2g} 2^{i+k} 3^{j+k} \binom{3g-2}{i} \binom{n-2g+2}{j} \binom{5g-3}{k}. \quad (7)$$

*Proof.* On the one hand,  $M_g(z) = \sum_{n \geq 2g} m_g(n) z^n$  since  $m_g(n) = 0$  if  $n < 2g$ . On the other hand, Theorem 1 gives  $P_g(m) = z^{-2g} F_g(m) M_g(z)$ , where  $z = m(1 - 3m)$ . From both of these results we obtain

$$P_g(m) = F_g(m) \sum_{n \geq 2g} m_g(n) (m(1 - 3m))^{n-2g}. \quad (8)$$

Since the degree of the polynomial  $P_g(m)$  is at most  $4g - 4$ ,  $P_g(m) = \sum_{l \leq 4g-4} p_{g,l} m^l$  with

$$p_{g,l} = [m^l] P_g(m) = \sum_{n \geq 2g}^{6g-4} m_g(n) [m^{l-n+2g}] (1 - 2m)^{3g-2} (1 - 3m)^{n-2g+2} (1 - 6m)^{5g-3}.$$

From  $(1 - am)^k = \sum_{i=0}^k \binom{k}{i} (-a)^i m^i$ , it follows that

$$p_{g,l} = \sum_{n=2g}^{6g-4} m_g(n) \sum_{i+j+k=l-n+2g} \binom{3g-2}{i} (-2)^i \binom{n-2g+2}{j} (-3)^j \binom{5g-3}{k} (-6)^k,$$

which implies Formula (7).  $\square$

The polynomial  $P_1(m)$  can be derived from [1] and the polynomials  $P_2(m)$  and  $P_3(m)$  from [4]. The polynomial  $P_4(m)$  was first given in [11]. All the polynomials  $P_g(m)$  with  $1 \leq g \leq 6$  can be found in [21, Appendix B]. They were computed by a complicated recurrence formula involving additional parameters. Theorem 3 and Carrell-Chapuy recurrence formula (3) provide a much more efficient way to compute the polynomials  $P_g(m)$ .

**Theorem 4.** For any positive integer  $g$ , the number  $m_g(n)$  of rooted maps of positive genus  $g$  with  $n$  edges is recursively defined for  $n \geq 6g - 3$  by

$$m_g(n) = \sum_{e=2g}^{n-1} (-1)^{n-e-1} m_g(e) \sum_{i+j+k=n-e} 2^{j+k} 3^{i+k} \binom{e-2g+2}{i} \binom{3g-2}{j} \binom{5g-3}{k}. \quad (9)$$

*Proof.* For  $n \geq 4g - 3$  it follows from (8) and the fact that the degree of the polynomial  $P_g(m)$  is at most  $4g - 4$  that

$$[m^n] \left( F_g(m) \sum_{e \geq 2g} m_g(e) (m(1 - 3m))^{e-2g} \right) = 0 \quad (10)$$

i.e.

$$\sum_{e=2g}^{n+2g} m_g(e) [m^{n-e+2g}] ((1 - 2m)^{3g-2} (1 - 3m)^{e-2g+2} (1 - 6m)^{5g-3}) = 0. \quad (11)$$

By isolating  $m_g(n+2g)$  in the left-hand side, we obtain

$$m_g(n+2g) = - \sum_{e=2g}^{n+2g-1} m_g(e) [m^{n-e+2g}] ((1-2m)^{3g-2} (1-3m)^{e-2g+2} (1-6m)^{5g-3}) \quad (12)$$

and Formula (9) follows from  $(1-am)^k = \sum_{i=0}^k \binom{k}{i} (-a)^i m^i$ .  $\square$

Theorems 3 and 4 also imply that the series  $(m_g(n))_{n \geq 0}$  of rooted maps of positive genus  $g$  is uniquely determined by its first  $6g-4$  values (among which the first  $2g$  values are known to be 0).

**Theorem 5.** *For any positive integer  $g$ , the number  $m_g(n)$  of rooted maps of positive genus  $g$  with  $n$  edges is defined by the closed formula*

$$m_g(n) = \sum_{l=0}^{4g-4} p_{g,l} \sum_{i+j+k=n-2g-l} 2^{i+k} 3^{j+k} \binom{i+3g-3}{i} \binom{j+n-2g+2}{j} \binom{k+5g-5}{k} \quad (13)$$

where the integers  $p_{g,l}$  are defined in Theorem 3.

*Proof.* By applying the Lagrange inversion formula [14, page 38] to (5), we obtain

$$[z^{n-2g}] \frac{M_g(z)}{z^{2g}} = [m^{n-2g}] \frac{P_g(m)(1-6m)}{F_g(m)(1-3m)^{n-2g+1}}$$

i.e.

$$m_g(n) = [m^{n-2g}] \frac{P_g(m)}{(1-2m)^{3g-2} (1-3m)^{n-2g+3} (1-6m)^{5g-4}}.$$

From  $(1-x)^{-p} = \sum_{k \geq 0} \binom{k+p-1}{k} x^k$  for  $p \geq 0$ , it follows that

$$m_g(n) = \sum_{l=0}^{4g-4} p_{g,l} \sum_{i+j+k=n-2g-l} \binom{i+3g-3}{i} 2^i \binom{j+n-2g+2}{j} 3^j \binom{k+5g-5}{k} 6^k$$

which implies Formula (13).  $\square$

As far as we know, these formulas are the first ones relating numbers of genus- $g$  rooted maps only with other numbers of rooted maps of the same genus. These formulas can be easily generalized to bivariate generating functions. We did not use them for counting unrooted maps because they are computationally less efficient than (3).

## 4 Counting unrooted maps

The second author has written a program in C++ that counts rooted maps by genus, number of edges and number of vertices using Formula (4) (an optimized version of [6, Corollary 3]) and also counts unrooted maps with these parameters. A table of numbers and a text file of the source code are available from the second author on request and in release 0.4.0 of the MAP project [9]. The source code can also be found in [20]. The method used to count unrooted maps given a table of numbers of rooted maps was presented by A. Mednykh and R. Nedela in [12] and refined by V. Liskovets in [10]. The second author used Liskovets' method. Details of the calculations were presented in [22]; a more pedagogical exposition can be found in [19]. For the sake of brevity we do not include the details here. Suffice it to say that the program uses the software CLN to handle big integers and the C/C++ compiler XCODE to run CLN; both of these software packages can be downloaded free of charge from the internet.

## 5 Complexity analysis and time trials

Let  $n$  be the largest number of edges in the maps to be enumerated. Then the maximum genus of the maps is  $\lfloor \frac{n}{2} \rfloor$ , since a map of genus  $g$  must have at least  $2g$  edges. The recurrence in [6] uses  $O(n^2)$  arithmetic operations to obtain the number of rooted maps with at most  $n$  edges, counted by number of edges alone, and  $O(n^3)$  arithmetic operations if one is counting by number of vertices as well as edges. To construct a table of numbers of rooted maps with up to  $n$  edges takes  $O(n^4)$  arithmetic operations if we are counting by number of edges alone, since there are  $O(n^2)$  entries in the table, and  $O(n^6)$  arithmetic operations if we are counting by number of vertices as well, since there are  $O(n^3)$  entries in the table.

These asymptotic estimates are overoptimistic if we are considering CPU time rather than number of arithmetic operations because we are working with big numbers. The arithmetic operation executed most often in counting rooted maps is multiplying one number of rooted maps by another one. The naïve method of multiplying two numbers of size (number of bits) bounded by  $b$  takes  $O(b^2)$  bit operations. When the numbers get big enough, CLN uses the Schönhage-Strassen method of multiplication [13], which uses  $O(b \log b \log \log b)$  bit operations. In [3] it was shown that the number of rooted maps of genus  $g$  with  $n$  edges is asymptotic to  $12n$  multiplied by a polynomial in  $n$  whose degree depends linearly on  $g$ . It follows that the number of bits in such a number is in  $O(n + g \log n)$ , and since  $g$  is in  $O(n)$ , the size of the number is in  $O(n \log n)$ . The true cost of counting rooted maps with up to  $n$  edges is found by multiplying the number of arithmetic operations by  $O(n^2(\log n)^2)$  if naïve multiplication is used and by  $O(n(\log n)^2 \log \log n)$  if the Schönhage-Strassen method is used.

The second author's program calculates  $m_g(n, f)$  for  $g$  going from 0 to a user-defined maximum, for  $n$  going from  $2g$  to a user-defined maximum and from  $f$  going from 1 to a maximum that makes  $v = n - f + 2(1 - g) = 1$  and reinterprets  $f$  as the number of vertices rather than the number of faces (by face-vertex duality). He conducted time trials on a portable Macintosh with a 2.66 GHz Intel Core 2 Duo processor, making  $n$  range from 20 to 100 in steps of 10 and setting the maximum value of  $g$  to be  $n/2$ . Here is the time in seconds for counting rooted maps for each of these values of  $n$  (0 means too small to be measured).

n	20	30	40	50	60	70	80	90	100
time	0	0	0.5	5	15	37	85	190	322

We do not have a time-complexity analysis for counting unrooted maps, but in the time trials it took less time to count unrooted maps than rooted ones. For  $n = 100$ , the time to count unrooted maps was 178 seconds.

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## Appendix: Numbers of unrooted maps

A table of numbers of unrooted planar maps (genus 0) with up to 6 edges can be found in [18]. Larger tables of numbers of unrooted planar maps were computed by N.C. Wormald and given privately to the second author, but as far as we know, they have never been published. A table of numbers of unrooted maps of genus 1-5 with up to 11 edges appears in [19] and [22]. We extend this table, both in terms of genus and number of edges, in this appendix.

The following table gives the numbers  $u_g(e, v)$  of genus- $g$  unrooted maps with  $e$  edges and  $v$  vertices, for  $g$  from 0 to 19 and for  $v$  from 1 to its maximal value  $e + 1 - 2g$ . The minimal value of  $e$  is  $2g$ . The maximal value of  $e$  is arbitrarily fixed so that the table fits five pages for genera 0 to 2, two pages for genera 3 to 5, and one page for higher genera. The maximal value for  $g$  is such that numbers fit in the page width.

$e$	$v$	$u_0(e, v)$	$u_1(e, v)$	$u_2(e, v)$
0	1	1		
0	sum	1		
1	1	1		
1	2	1		
1	sum	2		
2	1	1	1	
2	2	2		
2	3	1		
2	sum	4	1	
3	1	2	3	
3	2	5	3	
3	3	5		
3	4	2		
3	sum	14	6	
4	1	3	11	4
4	2	14	24	
4	3	23	11	
4	4	14		
4	5	3		
4	sum	57	46	4
5	1	6	46	53
5	2	42	180	53
5	3	108	180	
5	4	108	46	
5	5	42		
5	6	6		
5	sum	312	452	106



$e$	$v$	$u_0(e, v)$	$u_1(e, v)$	$u_2(e, v)$
6	1	14	204	553
6	2	140	1198	1276
6	3	501	2048	553
6	4	761	1198	
6	5	501	204	
6	6	140		
6	7	14		
6	sum	2071	4852	2382
7	1	34	878	4758
7	2	473	7212	18582
7	3	2264	18396	18582
7	4	4744	18396	4758
7	5	4744	7212	
7	6	2264	878	
7	7	473		
7	8	34		
7	sum	15030	52972	46680
8	1	95	3799	35778
8	2	1670	40776	205867
8	3	10087	142727	347558
8	4	27768	212443	205867
8	5	38495	142727	35778
8	6	27768	40776	
8	7	10087	3799	
8	8	1670		
8	9	95		
8	sum	117735	587047	830848
9	1	280	16304	244246
9	2	5969	219520	1910756
9	3	44310	999232	4747430
9	4	153668	2040348	4747430
9	5	279698	2040348	1910756
9	6	279698	999232	244246
9	7	153668	219520	
9	8	44310	16304	
9	9	5969		
9	10	280		
9	sum	967850	6550808	13804864
10	1	854	69486	1552834
10	2	21679	1139075	15680071
10	3	192444	6488604	52969260
10	4	816661	17227356	77948670
10	5	1873638	23634214	52969260
10	6	2458264	17227356	15680071
10	7	1873638	6488604	1552834

$e$	$v$	$u_0(e, v)$	$u_1(e, v)$	$u_2(e, v)$
10	8	816661	1139075	
10	9	192444	69486	
10	10	21679		
10	11	854		
10	sum	8268816	73483256	218353000
11	1	2694	294350	9349284
11	2	79419	5741220	117450580
11	3	828176	39779852	512308352
11	4	4200980	132209016	1025303224
11	5	11795964	235876296	1025303224
11	6	19509632	235876296	512308352
11	7	19509632	132209016	117450580
11	8	11795964	39779852	9349284
11	9	4200980	5741220	
11	10	828176	294350	
11	11	79419		
11	12	2694		
11	sum	72833730	827801468	3328822880
12	1	8714	1240308	53919954
12	2	293496	28271474	819971501
12	3	3537311	233068938	4452289504
12	4	21061347	942568684	11509375296
12	5	70719843	2105637162	15654660302
12	6	143157616	2738550608	11509375296
12	7	180492486	2105637162	4452289504
12	8	143157616	942568684	819971501
12	9	70719843	233068938	53919954
12	10	21061347	28271474	
12	11	3537311	1240308	
12	12	293496		
12	13	8714		
12	sum	658049140	9360123740	49325772812
13	1	28640	5202148	300331878
13	2	1091006	136580200	5412601192
13	3	15014328	1316388936	35599161080
13	4	103369288	6337310504	114602018272
13	5	407569560	17232289072	201379328048
13	6	986878680	28066908912	201379328048
13	7	1523077528	28066908912	114602018272
13	8	1523077528	17232289072	35599161080
13	9	986878680	6337310504	5412601192
13	10	407569560	1316388936	300331878
13	11	103369288	136580200	
13	12	15014328	5202148	
13	13	1091006		
13	14	28640		

$e$	$v$	$u_0(e, v)$	$u_1(e, v)$	$u_2(e, v)$
13	sum	6074058060	106189359544	714586880940
14	1	95640	21733696	1625426118
14	2	4078213	649405334	34132653009
14	3	63397256	7213525316	266220537080
14	4	498495378	40620565952	1038541797978
14	5	2273702888	131529397536	2273175492192
14	6	6466334844	260810488496	2936946412728
14	7	11939378311	326638072204	2273175492192
14	8	14615468757	260810488496	1038541797978
14	9	11939378311	131529397536	266220537080
14	10	6466334844	40620565952	34132653009
14	11	2273702888	7213525316	1625426118
14	12	498495378	649405334	
14	13	63397256	21733696	
14	14	4078213		
14	15	95640		
14	sum	57106433817	1208328304864	10164338225482
15	1	323396	90493272	8587132844
15	2	15312150	3046454992	207220225668
15	3	266509050	38537828328	1884416656912
15	4	2368459404	250230575696	8721265531848
15	5	12343172450	948078314200	23138230175172
15	6	40620147828	2239126384800	37241985748964
15	7	88106500004	3414411073976	37241985748964
15	8	129045594524	3414411073976	23138230175172
15	9	129045594524	2239126384800	8721265531848
15	10	88106500004	948078314200	1884416656912
15	11	40620147828	250230575696	207220225668
15	12	12343172450	38537828328	8587132844
15	13	2368459404	3046454992	
15	14	266509050	90493272	
15	15	15312150		
15	16	323396		
15	sum	545532037612	13787042250528	142403410942816
16	1	1105335	375691885	44442582224
16	2	57721030	14127535004	1218291353547
16	3	1116113327	201485902915	12739188485210
16	4	11110947214	1490633731778	68769605322980
16	5	65472242053	6514453678793	216512936399236
16	6	246254877247	18006841322290	422468300097440
16	7	618198141193	32708686628027	526326450852812
16	8	1063785332489	39826928417305	422468300097440
16	9	1272842946261	32708686628027	216512936399236
16	10	1063785332489	18006841322290	68769605322980
16	11	618198141193	6514453678793	12739188485210
16	12	246254877247	1490633731778	1218291353547
16	13	65472242053	201485902915	44442582224
16	14	11110947214	14127535004	

$e$	$v$	$u_0(e, v)$	$u_1(e, v)$	$u_2(e, v)$
16	15	1116113327	375691885	
16	16	57721030		
16	17	1105335		
16	sum	5284835906037	157700137398689	1969831979334086
17	1	3813798	1555771800	225971343444
17	2	218333832	64863745520	6968346176400
17	3	4658894160	1033998837648	82820994884096
17	4	51553861024	8628535594224	514298358102592
17	5	340432303072	42979642352848	1889177369500464
17	6	1448203830304	137064797207600	4375155072009488
17	7	4155977725664	291433805486672	6608420098046976
17	8	8278116804032	422739334207920	6608420098046976
17	9	11637788525696	422739334207920	4375155072009488
17	10	11637788525696	291433805486672	1889177369500464
17	11	8278116804032	137064797207600	514298358102592
17	12	4155977725664	42979642352848	82820994884096
17	13	1448203830304	8628535594224	6968346176400
17	14	340432303072	1033998837648	225971343444
17	15	51553861024	64863745520	
17	16	4658894160	1555771800	
17	17	218333832		
17	18	3813798		
17	sum	51833908183164	1807893066408464	26954132420126920
18	1	13269146	6428291934	1131367963884
18	2	828408842	295221527717	38919384594398
18	3	19391786118	5221086204768	520644158094148
18	4	236921843193	48720710849424	3676241660447931
18	5	1739717050754	273824061235756	15538149312306360
18	6	8295898355134	995529273862210	41992192647331392
18	7	26931885143228	2442526267219360	75288406812106052
18	8	61331742226722	4147624456667366	91282155067903038
18	9	99813869859301	4941186214175258	75288406812106052
18	10	117278995153034	4147624456667366	41992192647331392
18	11	99813869859301	2442526267219360	15538149312306360
18	12	61331742226722	995529273862210	3676241660447931
18	13	26931885143228	273824061235756	520644158094148
18	14	8295898355134	48720710849424	38919384594398
18	15	1739717050754	5221086204768	1131367963884
18	16	236921843193	295221527717	
18	17	19391786118	6428291934	
18	18	828408842		
18	19	13269146		
18	sum	514019531037910	20768681225892328	365393525753591368

$e$	$v$	$u_3(e, v)$	$u_4(e, v)$	$u_5(e, v)$
6	1	131		
6	sum	131		
7	1	4079		
7	2	4079		
7	sum	8158		
8	1	73282	14118	
8	2	167047		
8	3	73282		
8	sum	313611	14118	
9	1	970398	684723	
9	2	3693031	684723	
9	3	3693031		
9	4	970398		
9	sum	9326858	1369446	
10	1	10556722	17586433	2976853
10	2	58591595	39630698	
10	3	97799324	17586433	
10	4	58591595		
10	5	10556722		
10	sum	236095958	74803564	2976853
11	1	99944546	319763792	195644427
11	2	748976684	1192082898	195644427
11	3	1823736772	1192082898	
11	4	1823736772	319763792	
11	5	748976684		
11	6	99944546		
11	sum	5345316004	3023693380	391288854
12	1	852737424	4631706389	6623379011
12	2	8205279051	25016739573	14789629444
12	3	26989340556	41395800249	6623379011
12	4	39378084524	25016739573	
12	5	26989340556	4631706389	
12	6	8205279051		
12	7	852737424		
12	sum	111472798586	100692692173	28036387466
13	1	6709209232	56946090696	155182455738
13	2	79996972480	413223640688	569441291708
13	3	338043951088	991010148804	569441291708

$e$	$v$	$u_3(e, v)$	$u_4(e, v)$	$u_5(e, v)$
13	4	666422524608	991010148804	155182455738
13	5	666422524608	413223640688	
13	6	338043951088	56946090696	
13	7	79996972480		
13	8	6709209232		
13	sum	2182345314816	2922359760376	1449247494892
14	1	49461969282	617936108012	2841197873030
14	2	711640778177	5734881201032	15028479073373
14	3	3728403936278	18485468237252	24701811831354
14	4	9445619348392	26795029196244	15028479073373
14	5	12763979300656	18485468237252	2841197873030
14	6	9445619348392	5734881201032	
14	7	3728403936278	617936108012	
14	8	711640778177		
14	9	49461969282		
14	sum	40634231364914	76471600288836	60441165724160
15	1	345667110726	6074397541996	43425763829620
15	2	5878587435378	69634518493584	307366103788730
15	3	37184192378506	287198334481908	728458658338820
15	4	116833971177188	559637322350992	728458658338820
15	5	202918633990626	559637322350992	307366103788730
15	6	202918633990626	287198334481908	43425763829620
15	7	116833971177188	69634518493584	
15	8	37184192378506	6074397541996	
15	9	5878587435378		
15	10	345667110726		
15	sum	726322104184848	1845089145736960	2158501051914340
16	1	2310028835346	55099526091224	577374933310906
16	2	45675916449962	760174730620316	5209797503498640
16	3	341686270713324	3874407685623078	16527742407430762
16	4	1296601404482135	9662433645931070	23833896316372268
16	5	2793465994063884	12990353144165406	16527742407430762
16	6	3590596058829058	9662433645931070	5209797503498640
16	7	2793465994063884	3874407685623078	577374933310906
16	8	1296601404482135	760174730620316	
16	9	341686270713324	55099526091224	
16	10	45675916449962		
16	11	2310028835346		
16	sum	12550075287918360	41694584320696782	68463726004852884

$e$	$v$	$u_6(e, v)$	$u_7(e, v)$
12	1	1013582110	
12	sum	1013582110	
13	1	84928729933	
13	2	84928729933	
13	sum	169857459866	
14	1	3605028726801	508233789579
14	2	7992502487664	
14	3	3605028726801	
14	sum	15202559941266	508233789579
15	1	104340300511680	52147993673063
15	2	378134298777037	52147993673063
15	3	378134298777037	
15	4	104340300511680	
15	sum	964949198577434	104295987346126
16	1	2328771846722608	2680480846764174
16	2	12115285934958463	5908630695597199
16	3	19807266932574138	2680480846764174
16	4	12115285934958463	
16	5	2328771846722608	
16	sum	48695382495936280	11269592389125547
17	1	42879330119010060	92968027407241048
17	2	297515608385017712	333529278137138064
17	3	698425724113143808	333529278137138064
17	4	698425724113143808	92968027407241048
17	5	297515608385017712	
17	6	42879330119010060	
17	sum	2077641325234343160	852994611088758224
18	1	679574571686566150	2462686849706956592
18	2	5994737178116108922	12639396448986592872
18	3	18774176953946323287	20573712843056206498
18	4	26959977164375096074	12639396448986592872
18	5	18774176953946323287	2462686849706956592
18	6	5994737178116108922	
18	7	679574571686566150	
18	sum	77856954571873092792	50777879440443305426

$e$	$v$	$u_8(e, v)$	$u_9(e, v)$
16	1	352755124921122	
16	sum	352755124921122	
17	1	43058443920636593	
17	2	43058443920636593	
17	sum	86116887841273186	
18	1	2612103505736970587	324039613564554401
18	2	5730580864933991642	
18	3	2612103505736970587	
18	sum	10954787876407932816	324039613564554401
19	1	106104636805432131380	46037869184438374355
19	2	377468533878532051274	46037869184438374355
19	3	377468533878532051274	
19	4	106104636805432131380	
19	sum	967146341367928365308	92075738368876748710
20	1	3268090017604446925695	3231706843486368031963
20	2	16583906258119918465914	7061507183694710755564
20	3	26895334324381935980135	3231706843486368031963
20	4	16583906258119918465914	
20	5	3268090017604446925695	
20	sum	66599326875830666763353	13524920870667446819490
21	1	81763508749452267702334	151020126911739994806940
21	2	550484901682834216964372	533436706524721228557255
21	3	1273683419173516774041758	533436706524721228557255
21	4	1273683419173516774041758	151020126911739994806940
21	5	550484901682834216964372	
21	6	81763508749452267702334	
21	sum	3811863659211606517416928	1368913666872922446728390
22	1	1735799012483201542629310	5321407675084935890385252
22	2	14793365548485479622939589	26743956334292711312949466
22	3	45422458688847126828542788	43236837587662714557906902
22	4	64807620764474890716233843	26743956334292711312949466
22	5	45422458688847126828542788	5321407675084935890385252
22	6	14793365548485479622939589	
22	7	1735799012483201542629310	
22	sum	188710867264106506704457217	107367565606418008964576338



$e$	$v$	$u_{10}(e, v)$	$u_{11}(e, v)$
20	1	380751174738424280720	
20	sum	380751174738424280720	
21	1	61900350644739074439445	
21	2	61900350644739074439445	
21	sum	123800701289478148878890	
22	1	4950082376594691225742201	557175918657122229139987
22	2	10779106107210130980277396	
22	3	4950082376594691225742201	
22	sum	20679270860399513431761798	557175918657122229139987
23	1	262334467319926285470894622	102246856493968374607463423
23	2	920946297304483463377091334	102246856493968374607463423
23	3	920946297304483463377091334	
23	4	262334467319926285470894622	
23	sum	2366561529248819497695971912	204493712987936749214926846
24	1	10436807330236403989810496394	9197643937243060077009264642
24	2	52016489918140456428526385133	19968543728518640843922922089
24	3	83866835662326132082607669996	9197643937243060077009264642
24	4	52016489918140456428526385133	
24	5	10436807330236403989810496394	
24	sum	208773430159079852919281433050	38363831603004760997941451373
25	1	334074519898673454431872772378	546349421734820701894788862980
25	2	2200395535965333545033744718386	1907742061916852507697868346104
25	3	5037409011844160996918020411320	1907742061916852507697868346104
25	4	5037409011844160996918020411320	546349421734820701894788862980
25	5	2200395535965333545033744718386	
25	6	334074519898673454431872772378	
25	sum	15143758135416335992767275804168	4908182967303346419185314418168
26	1	8992412804931496094769804314194	24276552615926015429243306726942
26	2	74724657022260381172998180758989	120111902847763968111649111952181
26	3	226115216635966966996943064786668	193197660432676247606899872716724
26	4	321078006189133085356572309019684	120111902847763968111649111952181
26	5	226115216635966966996943064786668	24276552615926015429243306726942
26	6	74724657022260381172998180758989	
26	7	8992412804931496094769804314194	
26	sum	940742579115450773885994408739386	481974571360056214688684710074970

$e$	$v$	$u_{12}(e, v)$
24	1	993806827312044893602464496
24	sum	993806827312044893602464496
25	1	203568251472192593015565105153
25	2	203568251472192593015565105153
25	sum	407136502944385186031130210306
26	1	20384681425578629630065436540001
26	2	44139689150396597299844837950650
26	3	20384681425578629630065436540001
26	sum	84909052001553856559975711030652
27	1	1344032802282918564446470093660642
27	2	4670923997634519162591788341200373
27	3	4670923997634519162591788341200373
27	4	1344032802282918564446470093660642
27	sum	12029913599834875454076516869722030
28	1	66095585390798198366295835295463407
28	2	324909698939260325755902147746674575
28	3	521510766921112908280233047150555771
28	4	324909698939260325755902147746674575
28	5	66095585390798198366295835295463407
28	sum	1303521335581229956524629013234831735
29	1	2598647372085586327745995717592768700
29	2	16829619487924984024663810973279277072
29	3	38214110448983922489609206857083746964
29	4	38214110448983922489609206857083746964
29	5	16829619487924984024663810973279277072
29	6	2598647372085586327745995717592768700
29	sum	115284754617988985684038027095911585472
30	1	85392758017801624687683117609052737636
30	2	695858144741566474009926149792630696265
30	3	2081944949854120175333709476140932999277
30	4	2945354467291151637150567124347940326546
30	5	2081944949854120175333709476140932999277
30	6	695858144741566474009926149792630696265
30	7	85392758017801624687683117609052737636
30	sum	8671746172518128185213204611433173192902

$e$	$v$	$u_{13}(e, v)$
26	1	2122669454233128302149617542253
26	sum	2122669454233128302149617542253
27	1	480832429153352742558421356793665
27	2	480832429153352742558421356793665
27	sum	961664858306705485116842713587330
28	1	53130366325356854395291828009938727
28	2	114775986070991484071278106575375896
28	3	53130366325356854395291828009938727
28	sum	221036718721705192861861762595253350
29	1	3856356911137003381320301936446461074
29	2	13345600579548980014352553597690495308
29	3	13345600579548980014352553597690495308
29	4	3856356911137003381320301936446461074
29	sum	34403914981371966791345711068273912764
30	1	208265988956285322041288424854784582594
30	2	1017889232438589381522421291029823820959
30	3	1630734704264982296160323672383390044428
30	4	1017889232438589381522421291029823820959
30	5	208265988956285322041288424854784582594
30	sum	4083045147054731703287743104152606851534
31	1	8970722042074945753967336008761272150320
31	2	57686177099045026849910147518610059988352
31	3	130533817193066912073725928877428968894744
31	4	130533817193066912073725928877428968894744
31	5	57686177099045026849910147518610059988352
31	6	8970722042074945753967336008761272150320
31	sum	394381432668373769355206824809600602066832
32	1	322189870761730650250511467871042510892076
32	2	2603945884819023919715984436445634621702948
32	3	7753587725454287210291908773219679038077036
32	4	10951890382776236471789968498330693093538848
32	5	7753587725454287210291908773219679038077036
32	6	2603945884819023919715984436445634621702948
32	7	322189870761730650250511467871042510892076
32	sum	32311337344846320032306777853403405434882968

$e$	$v$	$u_{14}(e, v)$
28	1	5349362295912408418285480950292454
28	sum	5349362295912408418285480950292454
29	1	1329513645388215594553239451794715965
29	2	1329513645388215594553239451794715965
29	sum	2659027290776431189106478903589431930
30	1	160898561634069521892363237502437777425
30	2	346855611453502747893618316325861938456
30	3	160898561634069521892363237502437777425
30	sum	668652734721641791678344791330737493306
31	1	12765769708416431289559831195493899470210
31	2	44011104918179662694793003075316613282622
31	3	44011104918179662694793003075316613282622
31	4	12765769708416431289559831195493899470210
31	sum	113553749253192187968705668541621025505664
32	1	752086797680822688120006130623923595622946
32	2	3656730523358976901036890991989391463234050
32	3	5848430055307490353596054635418558343551050
32	4	3656730523358976901036890991989391463234050
32	5	752086797680822688120006130623923595622946
32	sum	14666064697387089531909848880645188461265042
33	1	35267296779456996454041460582407284988489428
33	2	225339610956320336710389271871310435524097264
33	3	508319887252694286639071826664484584536471066
33	4	508319887252694286639071826664484584536471066
33	5	225339610956320336710389271871310435524097264
33	6	35267296779456996454041460582407284988489428
33	sum	1537853589976943239607005118236404610098115516
34	1	1376188688409635874905853343655201243854733608
34	2	11039921104104953238907248969596220332376818700
34	3	32730965901080739246379167442830648082201463568
34	4	46166673137751515758000677859503539538189538108
34	5	32730965901080739246379167442830648082201463568
34	6	11039921104104953238907248969596220332376818700
34	7	1376188688409635874905853343655201243854733608
34	sum	136460824524942172478385217371667678855055569860

$e$	$v$	$u_{15}(e, v)$
30	1	15707315253480198543039354159336702543
30	sum	15707315253480198543039354159336702543
31	1	4254404066846916348883588403716819128103
31	2	4254404066846916348883588403716819128103
31	sum	8508808133693832697767176807433638256206
32	1	560291242973155478934912238951878579485370
32	2	1205555508775085796667646968133497238927983
32	3	560291242973155478934912238951878579485370
32	sum	2326137994721396754537471446037254397898723
33	1	48296285125745000564523254650901622578745046
33	2	165935861505220722116579262978988782927590263
33	3	165935861505220722116579262978988782927590263
33	4	48296285125745000564523254650901622578745046
33	sum	428464293261931445362205035259780811012670618
34	1	3085985802474307761706438171133166671774221272
34	2	14933910157176516402233695721551256357086244372
34	3	23848012464697728367174976612736644751185622064
34	4	14933910157176516402233695721551256357086244372
34	5	3085985802474307761706438171133166671774221272
34	sum	59887804383999376695055244398105490808906553352
35	1	156675496336534578862661113608020513244779757004
35	2	995278045846569420039029893381074747025407746934
35	3	223880580323545167737076841479360428504150689292
35	4	223880580323545167737076841479360428504150689292
35	5	995278045846569420039029893381074747025407746934
35	6	156675496336534578862661113608020513244779757004
35	sum	6781518690837298333277535696936911377548676386460
36	1	6607804650427895027651495257432388201136957637604
36	2	52651158391352383107127670390450702272652846669207
36	3	155486533895708212045879189159875151451391021408966
36	4	219028932036157434744940729921104818730252539947224
36	5	155486533895708212045879189159875151451391021408966
36	6	52651158391352383107127670390450702272652846669207
36	7	6607804650427895027651495257432388201136957637604
36	sum	648519925911134415106257439536621302580614191378778

$e$	$v$	$u_{16}(e, v)$
32	1	53160783752637968542926390100726167551610
32	sum	53160783752637968542926390100726167551610
33	1	15600320175507300729043193735546421507235073
33	2	15600320175507300729043193735546421507235073
33	sum	31200640351014601458086387471092843014470146
34	1	2223274627983595422507340640133497482768944961
34	2	4775494679658728042571008949941678370459941602
34	3	2223274627983595422507340640133497482768944961
34	sum	9222043935625918887585690230208673335997831524
35	1	207098988317023096921058787758993781463514981524
35	2	709331542316073084891616565646314491079275188466
35	3	709331542316073084891616565646314491079275188466
35	4	207098988317023096921058787758993781463514981524
35	sum	1832861061266192363625350706810616545085580339980
36	1	14279273921214976669647789455348595104544279637003
36	2	68805165020043702416786649792601959818097931393304
36	3	109721245211691855532004451836827883534451504403689
36	4	68805165020043702416786649792601959818097931393304
36	5	14279273921214976669647789455348595104544279637003
36	sum	275890123094209213704873330332728993379735926464303
37	1	781105068702241754910796845876629366053531576061370
37	2	4935719985964953802862159423860034674318681677247312
37	3	11073916808061482567461763466695337023784694521589900
37	4	11073916808061482567461763466695337023784694521589900
37	5	4935719985964953802862159423860034674318681677247312
37	6	781105068702241754910796845876629366053531576061370
37	sum	33581483725457356250469439472864002128313815549797164
38	1	35441207898429792043123436123187064049015449552488250
38	2	280656201598016620390674260860638474559649068763638975
38	3	825842658906095001495942350546153309388786108632837332
38	4	1161964050508679879461554362247938069968931276241633907
38	5	825842658906095001495942350546153309388786108632837332
38	6	280656201598016620390674260860638474559649068763638975
38	7	35441207898429792043123436123187064049015449552488250
38	sum	3445844187313762707321034457307895765963832530139563021

$e$	$v$	$u_{17}(e, v)$
34	1	205445432928009832581491069006516880609705841
34	sum	205445432928009832581491069006516880609705841
35	1	64986432002837812745875711198508285517101944963
35	2	64986432002837812745875711198508285517101944963
35	sum	129972864005675625491751422397016571034203889926
36	1	9973161581095456726301922365556925358027314253479
36	2	21388203433438836924480615057329648012547524988820
36	3	9973161581095456726301922365556925358027314253479
36	sum	41334526595629750377084459788443498728602153495778
37	1	999214401417939130045637235023670174633866987201502
37	2	3412631887102397881486610278438413589213538056913876
37	3	3412631887102397881486610278438413589213538056913876
37	4	999214401417939130045637235023670174633866987201502
37	sum	8823692577040674023064495026924167527694810088230756
38	1	74008165218025656228403282401053203578535203947902856
38	2	355210215550916519986577506055980731891976679082990264
38	3	565713794642371539921830564202098813295282815677680430
38	4	355210215550916519986577506055980731891976679082990264
38	5	74008165218025656228403282401053203578535203947902856
38	sum	1424150556180255892351792141116166684236306581739466670
39	1	4343202278169258593275684010568229749944220951468232434
39	2	27311052025745996059683391861883339932687540212345629528
39	3	61130753148022353058797706674786842441835799506625894124
39	4	61130753148022353058797706674786842441835799506625894124
39	5	27311052025745996059683391861883339932687540212345629528
39	6	4343202278169258593275684010568229749944220951468232434
39	sum	185570014903875215423513565094476824248935121340879512172
40	1	211137508713213986158094387442923550756912980107062330239
40	2	1662514046685373766481849147320976176363162317404644178392
40	3	4875869143426687082059537470267033644594639078062839477983
40	4	6852916545721885675371996110251583758639955478551923500331
40	5	4875869143426687082059537470267033644594639078062839477983
40	6	1662514046685373766481849147320976176363162317404644178392
40	7	211137508713213986158094387442923550756912980107062330239
40	sum	20351957943372435344770958120313450502069384229701015473559

$e$	$v$	$u_{18}(e, v)$
36	1	899180059563093845406786676951930933700666538428
36	sum	899180059563093845406786676951930933700666538428
37	1	305209945591860728601292564295170264716895126366229
37	2	305209945591860728601292564295170264716895126366229
37	sum	610419891183721457202585128590340529433790252732458
38	1	50219426896416777538218063589516219785265760267129801
38	2	107543707445670286349292801498822861069642312371491692
38	3	50219426896416777538218063589516219785265760267129801
38	sum	207982561238503841425728928677855300640173832905751294
39	1	5389202115814743871428660271901529819121489449229108268
39	2	18357522907485928372236446971581267536474065605106694205
39	3	18357522907485928372236446971581267536474065605106694205
39	4	5389202115814743871428660271901529819121489449229108268
39	sum	47493450046601344487330214486965594711191110108671604946
40	1	427068485931184787066302658812981942725804369479622658290
40	2	2042346003975213807545249105132487353129138259822245439027
40	3	3248824444658530733670030530464329613423303443228840360174
40	4	2042346003975213807545249105132487353129138259822245439027
40	5	427068485931184787066302658812981942725804369479622658290
40	sum	8187653424471327922893134058355268205133188701832576554808
41	1	26784645725041818334522973443000470040238838410719829307538
41	2	167674309375247041644579797461104409791966365092985054355672
41	3	374488402865205525083259282243771736446866572624028195496896
41	4	374488402865205525083259282243771736446866572624028195496896
41	5	167674309375247041644579797461104409791966365092985054355672
41	6	26784645725041818334522973443000470040238838410719829307538
41	sum	1137894715930988770124724106295753232558143552255466158320212
42	1	1389944269863838497811621155190654489220344748250650534194358
42	2	10887331081526501822853177759545887968080128197675407799362682
42	3	31833348001308959952129406291954071592650439711678587040169837
42	4	44696120539232218172253544414937593313423676099859800982308502
42	5	31833348001308959952129406291954071592650439711678587040169837
42	6	10887331081526501822853177759545887968080128197675407799362682
42	7	1389944269863838497811621155190654489220344748250650534194358
42	sum	132917367244630818717841954828318821413325501415069091729762256



$e$	$v$	$u_{19}(e, v)$
38	1	4424730378121305321456186121529010463964830910484003
38	sum	4424730378121305321456186121529010463964830910484003
39	1	1605186831344690925467081160430702520300496445763873311
39	2	1605186831344690925467081160430702520300496445763873311
39	sum	3210373662689381850934162320861405040600992891527746622
40	1	282085119538112569201481233917466805746825156704653052178
40	2	603272708788070683140741539446707320754385893163661763443
40	3	282085119538112569201481233917466805746825156704653052178
40	sum	1167442947864295821543704007281640932248036206572967867799
41	1	32302728422532346841074282111005042311763117883791879554812
41	2	109767667148746367413775667829043306525555960548263837730600
41	3	109767667148746367413775667829043306525555960548263837730600
41	4	32302728422532346841074282111005042311763117883791879554812
41	sum	284140791142557428509699899880096697674638156864111434570824
42	1	2729008813037064165588608037007491955895014418138507486290090
42	2	13007112201580570259600784438565181934710759194453710021264759
42	3	20668163895282079484958851224259658246014568460489861510080658
42	4	13007112201580570259600784438565181934710759194453710021264759
42	5	2729008813037064165588608037007491955895014418138507486290090
42	sum	52140405924517348335337636175405006027226115685674296525190356
43	1	182285420872810479366334700885431830095294173825146404386445914
43	2	1136394999010390323212649937280709365011343445435404008508674492
43	3	2532921941159641534900117827607810131249613553947035703033638132
43	4	2532921941159641534900117827607810131249613553947035703033638132
43	5	1136394999010390323212649937280709365011343445435404008508674492
43	6	182285420872810479366334700885431830095294173825146404386445914
43	sum	7703204722085684674958204931547902652712502346415172231857517076
44	1	10064181576960495335927932858268075358326289452842097820366680160
44	2	78450121794974638316938365981758852583573376009308176509400884718
44	3	228731201293361297266257064521272334489573226866484237764617340414
44	4	320854956051603947387522506512966561014063478810265368757633296296
44	5	228731201293361297266257064521272334489573226866484237764617340414
44	6	78450121794974638316938365981758852583573376009308176509400884718
44	7	10064181576960495335927932858268075358326289452842097820366680160
44	sum	955345965382196809225769233235565085877009263467534392946403106880